The question of the number of zeroes at the end of $N!$ is a chestnut dating back at least several decades. The well-known answer is

$$L_5(N) = \sum_{i=1}^{\infty} \lfloor N/5^i \rfloor.$$

Justification: There is exactly one trailing zero for each factor of 10 in $N!$, and each factor of 10 arises from a factor of 2 and a factor of 5. There are obviously many more of the former, so we need only count factors of 5. Every fifth integer contributes a factor of 5, total $\lfloor N/5 \rfloor$. Every twenty-fifth integer contributes another, for an additional $\lfloor N/5^2 \rfloor$, and so forth.

Now let's ask what happens when $N!$ is written in a base other than 10. Let $Z_b(N)$ denote the number of trailing zeroes when $N!$ is written in integral base $b \geq 2$. Once again, each such zero arises from a factor of b. Writing b as its unique prime factorization $b = 2^{b_2} 3^{b_3} \cdots p_i^{b_{p_i}} \cdots$ where p_i is the i^{th} prime, we see that to get a trailing zero we need b_2 factors of 2, b_3 factors of 3, and so forth. The number of factors of p in $N!$ is $L_p(N)$, so each p with $b_p > 0$ constrains the number of trailing zeros to be at most $\lfloor L_p(N)/b_p \rfloor$. The number we want is the minimum of all these constraints:

$$Z_b(N) = \min\{\lfloor L_{p_i}(N)/b_{p_i} \rfloor\} \tag{1}$$

where the minimum is taken over all i such that p_i divides b.

For $b = 10$ this formula becomes $\min\{L_2(N), L_5(N)\}$. But we've said that the answer is simply $L_5(N)$ because there are "obviously" many more factors of 2 than of 5. That is, 5 is the "limiting factor"; the L_2 term can be ignored because it's always greater than the L_5 term. In the rest of this note we examine this question more generally, asking when exactly can we ignore terms of (1) as we can ignore the L_2 term when $b = 10$.

We start by generalizing the observation that worked for $b = 10$: If p and q are positive integers with $p < q$, then clearly $L_p(N) \geq L_q(N)$. Hence when b has prime factors p and q with $p < q$ and $b_p \leq b_q$ there will always be enough factors of p to go around, that is, $\lfloor L_p(N)/b_p \rfloor$ is necessarily greater than $\lfloor L_q(N)/b_q \rfloor$ and the L_p term in (1) can be ignored.

So, for example, with $b = 13500 = 2^2 3^3 5^3$ we have $Z_b(N) = \lfloor Z_5(N)/3 \rfloor$ since the L_2 and L_3 terms in the minimum are necessarily larger than the L_5 term. Similarly, with $b = 1389150 = 2^1 3^4 5^2 7^3$ we can ignore the L_2 and L_5 terms, but not the others, giving $Z_b(N) = \min\{\lfloor L_3(N)/4 \rfloor, \lfloor L_7(N)/3 \rfloor\}$.

But this can't be the whole story. Consider base $b = 3072 = 2^{10}3^1$. Clearly 2 is now the limiting factor: we need so many 2s for each 3 that there will always be 3s and to spare. Hence we need only count 2s, that is, we can ignore the L_3 term in (1). The rule above doesn't capture this case.

Let's estimate the value of $\lfloor Z_p(N)/b_p \rfloor$ by dropping all use of the floor function. The result is $N/(p-1)b_p$ since Z_p becomes simply the sum of a geometric sequence. So $1/(p-1)b_p$ is a multiplier that approximates the proportion of trailing zeros that p can contribute to $N!$. Now when b has factors p^{b_p} and q^{b_q}, the smaller of $1/(p-1)b_p$ and $1/(q-1)b_q$ indicates the limiting factor, and we can drop the term in (1) corresponding to the larger. It's easier to work with reciprocals, retaining the term corresponding to the *larger* of $(p-1)b_p$ and $(q-1)b_q$. In the $b = 3072$ example, $(2-1)10$ is greater than $(3-1)1$, hence we can discard the L_3 term of (1), as we had already concluded. Note that this rule subsumes the previous rule, since if $p < q$ and $b_p < b_q$ then necessarily $(q-1)b_q > (p-1)b_q$.

The correctness of this new rule depends on the following lemma: For primes p and q and positive integers b_p and b_q such that $(q-1)b_q > (p-1)b_p$, we have $\lfloor Z_p(N)/b_p \rfloor \geq \lfloor Z_q(N)/b_q \rfloor$ for all N. Is this lemma true?

It's easy to see that it's true for all sufficiently large N, that is, for any such p, q, b_p, b_q there exists N_0 such that the lemma is true for all $N > N_0$. Proof: Dropping a single use of the floor function increases a value by at most 1, so dropping all floor functions in $L_p(N)$ increases its value by at most $\log_p N$ plus a constant. Hence our estimate is too high by at most $(\log_p N)/b_p$ plus a constant. But the difference between two such estimates is N times a fixed constant, and so for all sufficiently large N exceeds the maximum possible error. Hence using the estimate yields the correct answer for large N. Based on numerical experimentation we conjecture that the lemma is in fact true for all N.

Finally, what happens when $(p-1)b_p$ is exactly equal to $(q-1)b_q$? Base $b = 12 = 2^2 3^1$ is a simple example: $(2-1)*2 = (3-1)*1 = 2$. Is there an additional criterion we can use to eliminate one term or the other? We conjecture that the answer is no, that is, we believe that for primes p and q and positive integers b_p and b_q such that $(p-1)b_p = (q-1)b_q$, we have $\lfloor Z_p(N)/b_p \rfloor > \lfloor Z_q(N)/b_q \rfloor$ for infinitely many N. (This conjecture is borne out by experimental evidence, but we have no other justification for it.) It follows that if there are two terms with maximal $(p-1)b_p$, both must be retained. Assuming the truth of the conjecture generalized in the obvious way to multiple primes, all terms with maximal $(p-1)b_p$ must be retained.

In summary, if our conjectures are true, then the terms of (1) that may be discarded are exactly those whose value of $(p_i - 1)b_{p_i}$ is not maximal.